Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 307–323
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Hermitian–Walker 4-manifolds

J. Davidov[a], J.C. Díaz-Ramos[b], E. García-Río[c,*], Y. Matsushita[d], O. Muškarov[a], R. Vázquez-Lorenzo[c]

[a] *Institute of Mathematics and Informatics, Bulgarian Academy of Science, Sofia, Bulgaria*
[b] *Department of Mathematics, University College Cork, Cork, Ireland*
[c] *Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain*
[d] *Section of Mathematics, School of Engineering, University of Shiga Prefecture, Hikone 522-8533, Japan*

Received 22 June 2007; received in revised form 9 November 2007; accepted 18 November 2007
Available online 22 November 2007

Abstract

A local description of proper Hermitian–Walker structures that are locally conformally Kähler, self-dual, $*$-Einstein or Einstein is given. This is used to supply examples of indefinite Einstein strictly almost Hermitian structures showing that an integrability result in [K.-D. Kirchberg, Integrability conditions for almost Hermitian and almost Kähler 4-manifolds, arXiv:math.DG/0605611] does not hold for metrics of signature (2, 2).

MSC: 53C50; 53B30

Keywords: Walker metric; Proper almost complex structure; Hermitian structure; Locally conformally Kähler structure; Einstein; $*$-Einstein; Self-dual metric

1. Introduction

An almost Hermitian structure on a smooth manifold M consists of an almost complex structure J and a metric g satisfying the compatibility condition $g(JX, JY) = g(X, Y)$. If the almost complex structure J is integrable (i.e. comes from a complex structure on M), the structure (g, J) is called Hermitian. Any almost Hermitian structure (g, J) determines a non-degenerate 2-form $\Omega(X, Y) = g(JX, Y)$. If Ω is closed (i.e., it is a symplectic form) the structure is said to be almost Kähler and (g, J) is called Kähler if, in addition, the almost complex structure J is integrable.

A basic problem in almost Hermitian geometry is to relate properties of the structure (g, J) to the curvature of (M, g). For example, the well-known Goldberg conjecture [14] claims that a compact almost Kähler manifold is Kähler provided the metric g is Einstein. This conjecture was proved by Sekigawa [26] in the case of non-negative scalar curvature but it is still far from being solved in the negative case. We refer to the survey [2] for an update on the

* Corresponding author.
E-mail addresses: jtd@math.bas.bg (J. Davidov), JC.DiazRamos@ucc.ie (J.C. Díaz-Ramos), xtedugr@usc.es (E. García-Río), yasuo@mech.usp.ac.jp (Y. Matsushita), muskarov@math.bas.bg (O. Muškarov), ravazlor@usc.es (R. Vázquez-Lorenzo).

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.11.006

integrability of almost Kähler structures. Another integrability result related to the curvature properties of a manifold is the Riemannian version of the well-known Goldberg–Sacks theorem in General Relativity. It says that an oriented Einstein 4-manifold admits locally a compatible complex structure if and only if the spectrum of the positive Weyl tensor is degenerate [23,25]. We refer to [3] for generalizations of this result in the Riemannian setting and to [1] for analogous results for arbitrary pseudo-Riemannian 4-manifolds.

One should note that some integrability results for almost Hermitian manifolds are not true in the case of indefinite metrics. For example, it is shown in [8] that, in contrast to the Riemannian case, there are local examples of flat non-Kähler almost Kähler metrics of signature (2, 2). Moreover, an indefinite Ricci flat strictly almost Kähler metric on eight-dimensional torus has been recently reported in [22].

The purpose of this paper is to provide a large family of non-Kähler isotropic Kähler–Hermitian structures having interesting curvature properties. To this end we consider Walker metrics [27] on 4-manifolds together with the so-called proper almost complex structure [21] and obtain a local description of those metrics which are Hermitian or locally conformally Kähler and self-dual, $*$-Einstein or Einstein. We also construct examples of indefinite Einstein strictly almost Hermitian structures showing that an integrability result in [19] does not hold for metrics of signature (2, 2).

2. Preliminaries

Throughout this paper we use the following convention for the curvature tensor $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, where ∇ denotes the Levi-Civita connection. $\rho(X, Y) = \mathrm{trace}\,\{U \rightsquigarrow R(X, U)Y\}$ and $\tau = \mathrm{trace}\,\rho$ are the Ricci tensor and the scalar curvature, respectively. As usual, (M, g) is said to be Einstein if $\rho = \frac{\tau}{n} g$, $n = \dim M$, in which case the scalar curvature is necessarily constant.

Let M be an oriented four-dimensional manifold with a neutral metric g, i.e. a metric of signature (2, 2). The metric g induces an inner product on the bundle Λ^2 of bivectors by

$$\langle X_1 \wedge X_2, X_3 \wedge X_4\rangle = g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3),$$

$X_1, \dots, X_4 \in TM$. Let $\mathbf{e}_1, \dots, \mathbf{e}_4$ be a local oriented orthonormal frame of TM with $\|\mathbf{e}_1\|^2 = \|\mathbf{e}_2\|^2 = 1$, $\|\mathbf{e}_3\|^2 = \|\mathbf{e}_4\|^2 = -1$. As in the Riemannian case, the Hodge star operator $* : \Lambda^2 \to \Lambda^2$ is an involution given by

$$*(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \wedge \mathbf{e}_4, \qquad *(\mathbf{e}_1 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_4, \qquad *(\mathbf{e}_1 \wedge \mathbf{e}_4) = -\mathbf{e}_2 \wedge \mathbf{e}_3.$$

Denote by $\Lambda_\pm$ the subbundles of Λ^2 determined by the eigenvalues ± 1 of the Hodge star operator. Set

$$\begin{aligned} s_1 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4), & \bar{s}_1 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4), \\ s_2 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_4), & \bar{s}_2 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4), \\ s_3 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3), & \bar{s}_3 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3). \end{aligned} \tag{1}$$

Then $\{s_1, s_2, s_3\}$ and $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ are local oriented orthonormal frames of Λ_- and Λ_+, respectively, with $\|s_1\|^2 = \|\bar{s}_1\|^2 = 1$, $\|s_2\|^2 = \|\bar{s}_2\|^2 = \|s_3\|^2 = \|\bar{s}_3\|^2 = -1$.

Further we shall often identify Λ^2 with the bundle of skew-symmetric endomorphisms of TM by the correspondence that assigns to each $\sigma \in \Lambda^2$ the endomorphism K_σ on T_pM, $p = \pi(\sigma)$, defined by

$$g(K_\sigma X, Y) = g(\sigma, X \wedge Y); \quad X, Y \in T_pM. \tag{2}$$

Considering the Riemann curvature tensor as an endomorphism of Λ^2, we have the following $SO(2, 2)$-decomposition

$$R \equiv \frac{\tau}{12} Id_{\Lambda^2} + \rho_0 + W^+ + W^- : \Lambda^2 \to \Lambda^2, \tag{3}$$

where τ is the scalar curvature, ρ_0 denotes the traceless Ricci tensor, $W = W^+ + W^-$ is the Weyl conformal curvature tensor and $W^\pm = \frac{1}{2}(W \pm *W)$. Recall that a pseudo-Riemannian 4-manifold is called *self-dual* (resp., *anti-self-dual*) if $W^- = 0$ (resp., $W^+ = 0$).

A *Walker manifold* is a triple $(M, g, \mathcal{D})$ where M is an n-dimensional manifold, g an indefinite metric and $\mathcal{D}$ an r-dimensional parallel null distribution. Of special interest are those manifolds admitting a field of null planes of maximal dimension $r = \frac{n}{2}$. Since the dimension of a null plane is $r \le \frac{n}{2}$, the lowest possible case of a Walker metric is that of $(++--)$-manifolds admitting a field of parallel null two-planes.

The Walker metrics appear in several specific pseudo-Riemannian structures like 2-step nilpotent Lie groups with degenerate center, parakähler and hyper-symplectic structures, hypersurfaces with nilpotent shape operator and some four-dimensional Osserman manifolds. Indecomposable metrics of neutral signature (playing a distinguished role in investigating holonomy of indefinite metrics) are also equipped with a Walker structure. This clearly motivates the study of pseudo-Riemannian manifolds carrying a parallel degenerate plane field (see [8] and the references therein for more information).

For our purposes it is convenient to use special coordinate systems associated to any Walker metric. Recall that, by a result of Walker [27], for every Walker metric g on a 4-manifold M, there exist local coordinates (x, y, z, t) around any point of M such that the matrix of g in these coordinates has the following form

$$g_{(x,y,z,t)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix} \tag{4}$$

for some functions a, b and c depending on the coordinates (x, y, z, t). As a matter of notation, throughout this work we denote by ∂_i the coordinate tangent vectors, $i = x, \dots, t$. Also, h_i means partial derivative $\frac{\partial h}{\partial i}$, $i = x, \dots, t$, for any function $h(x, y, z, t)$. Expressions for the Levi-Civita connection, curvature tensor and Ricci tensor of a Walker metric (4) are available in [8,9,21]. We omit the details for the sake of brevity.

3. Proper almost hyper-parahermitian structures

An almost hyper-paracomplex structure on a $4n$-dimensional manifold M is a triple (J_1, J_2, J_3), where J_2, J_3 are almost paracomplex structures (cf. [16]) and J_1 is an almost complex structure, satisfying the paraquaternionic identities

$$J_1^2 = -J_2^2 = -J_3^2 = -1, \qquad J_1 J_2 = -J_2 J_1 = J_3.$$

A hyper-parahermitian metric is a pseudo-Riemannian metric which is compatible with the (almost) hyper-paracomplex structure in the sense that the metric g is skew-symmetric with respect to each J_i, $i = 1, 2, 3$, i.e.

$$g(J_1 \cdot, J_1 \cdot) = -g(J_2 \cdot, J_2 \cdot) = -g(J_3 \cdot, J_3 \cdot) = g(\cdot, \cdot).$$

Such a structure is called hyper-parahermitian if all the structures J_i are integrable. If each J_i, $i = 1, 2, 3$, is parallel with respect to the Levi-Civita connection or, equivalently, the three Kähler forms $\Omega_i(X, Y) = g(J_i X, Y)$ are closed, then the manifold is called hyper-symplectic [15] or hyper-parakählerian. In this case J_2 and J_3 are parakähler structures and it follows that g is a Walker metric (see [17,18] for more information).

If (g, J_1, J_2, J_3) is an almost hyper-parahermitian structure, then the bivectors corresponding via the metric to the 2-forms Ω_1, Ω_2, Ω_3 define an orthonormal basis of Λ^2_-, and conversely, any orthonormal basis of Λ^2_- defines an almost hyper-parahermitian structure.

Let g be a Walker metric on $\mathbb{R}^4$ having the form (4). Then an orthonormal frame of $T\mathbb{R}^4$ can be specialized by using the canonical coordinates as follows:

$$\begin{aligned} \mathbf{e}_1 &= \frac{1-a}{2}\partial_x + \partial_z, \qquad & \mathbf{e}_2 &= \frac{1-b}{2}\partial_y + \partial_t - c\partial_x, \\ \mathbf{e}_3 &= -\frac{1+a}{2}\partial_x + \partial_z, \qquad & \mathbf{e}_4 &= -\frac{1+b}{2}\partial_y + \partial_t - c\partial_x. \end{aligned} \tag{5}$$

Let $\{s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3\}$ be the frame of $\Lambda^2 = \Lambda_- \oplus \Lambda_+$ defined by means of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ via (1). Then

$$\begin{aligned} s_1 &= \frac{1}{\sqrt{2}}\left(-\frac{a+b}{2}\partial_x \wedge \partial_y + \partial_x \wedge \partial_t - \partial_y \wedge \partial_z\right), \\ s_2 &= \frac{1}{\sqrt{2}}(\partial_x \wedge \partial_z - \partial_y \wedge \partial_t - c\partial_x \wedge \partial_y), \\ s_3 &= \frac{1}{\sqrt{2}}\left(\frac{a-b}{2}\partial_x \wedge \partial_y + \partial_x \wedge \partial_t + \partial_y \wedge \partial_z\right), \end{aligned} \tag{6}$$

and

$$\begin{aligned} \bar{s}_1 &= \frac{1}{\sqrt{2}}\left(\frac{1+ab}{2}\partial_x \wedge \partial_y + 2c\partial_x \wedge \partial_z - a\partial_x \wedge \partial_t + b\partial_y \wedge \partial_z + 2\partial_z \wedge \partial_t\right), \\ \bar{s}_2 &= \frac{1}{\sqrt{2}}(c\partial_x \wedge \partial_y + \partial_x \wedge \partial_z + \partial_y \wedge \partial_t), \\ \bar{s}_3 &= \frac{1}{\sqrt{2}}\left(\frac{ab-1}{2}\partial_x \wedge \partial_y + 2c\partial_x \wedge \partial_z - a\partial_x \wedge \partial_t + b\partial_y \wedge \partial_z + 2\partial_z \wedge \partial_t\right). \end{aligned} \tag{7}$$

The bivectors s_1, s_2, s_3 define via (2) endomorphisms J_1, J_2, J_3 of $T\mathbb{R}^4$ such that

$$J_1^2 = -1, \qquad J_2^2 = J_3^2 = 1, \qquad J_1 J_2 = -J_2 J_1 = J_3.$$

We shall say that the almost hyper-paracomplex structure J_1, J_2, J_3 defined by means of s_1, s_2, s_3 is *proper*. Note that J_1 is an isometry of the Walker metric g while J_2, J_3 are anti-isometries, i.e. J_1, J_2, J_3 is an almost hyper-parahermitian structure.

Recall that an indefinite almost Hermitian (resp., almost parahermitian) structure (g, J) is said to be *isotropic Kähler* (resp., *parakähler*) if $\|\nabla J\|^2 = 0$. Isotropic Kähler structures were first investigated in [13] in dimension four and subsequently in [4] in dimension six. It has been shown by the authors in [8] that any proper almost Hermitian structure is isotropic Kähler. Moreover, we have the following:

Theorem 1. *Any proper almost hyper-parahermitian structure* (g, J_1, J_2, J_3) *on a Walker 4-manifold satisfies* $\|\nabla J_i\|^2 = 0$, $\|\mathrm{d}\Omega_i\|^2 = 0$, $\|\delta\Omega_i\|^2 = 0$ *and* $\|N_{J_i}\|^2 = 0$, *where* Ω_i *denotes the fundamental 2-form and* N_{J_i} *the Nijenhuis tensor of* J_i, $i = 1, 2, 3$.

Proof. From the expressions (4) and (6) one gets the description in local coordinates of the structures J_i $(i = 1, 2, 3)$ as follows:

$$J_1 = \begin{pmatrix} 0 & -1 & -c & \frac{a-b}{2} \\ 1 & 0 & \frac{a-b}{2} & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & -1 & 0 & -b \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & 1 & c & \frac{a+b}{2} \\ 1 & 0 & \frac{a+b}{2} & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Now the result follows after some calculations as in [8]. □

Theorem 2. *A proper almost hyper-parahermitian structure* J_1, J_2, J_3 *is hyper-parahermitian (i.e.* J_1, J_2, J_3 *are integrable) if and only if the functions a, b and c in* (4) *have the form*

$$\begin{aligned} a &= x^2\mathcal{B}(z,t) + xP(z,t) + \xi(z,t), \\ b &= y^2\mathcal{B}(z,t) + yT(z,t) + \eta(z,t), \\ c &= xy\mathcal{B}(z,t) + \frac{1}{2}xT(z,t) + \frac{1}{2}yP(z,t) + \gamma(z,t), \end{aligned} \tag{8}$$

where the capital, calligraphic and Greek letters stand for arbitrary smooth functions depending only on the coordinates (z, t).

Proof. Let $\Omega_i(X,Y) = g(J_i X, Y)$ be the fundamental 2-form of (g, J_i), $i = 1,2,3$. Then we have

$$\Omega_1 = \mathrm{d}x \wedge \mathrm{d}t - \mathrm{d}y \wedge \mathrm{d}z + \frac{a+b}{2}\mathrm{d}z \wedge \mathrm{d}t,$$
$$\Omega_2 = -\mathrm{d}x \wedge \mathrm{d}z + \mathrm{d}y \wedge \mathrm{d}t + c\mathrm{d}z \wedge \mathrm{d}t,$$
$$\Omega_3 = -\mathrm{d}x \wedge \mathrm{d}t - \mathrm{d}y \wedge \mathrm{d}z - \frac{a-b}{2}\mathrm{d}z \wedge \mathrm{d}t.$$

The differential of Ω_i has the form $\mathrm{d}\Omega_i = \omega_i \wedge \Omega_i$, $i = 1,2,3$, where the Lee forms ω_i are given by

$$\begin{aligned} \omega_1 &= -\frac{1}{2}(a+b)_x \mathrm{d}z - \frac{1}{2}(a+b)_y \mathrm{d}t, \\ \omega_2 &= -c_y \mathrm{d}z - c_x \mathrm{d}t, \\ \omega_3 &= -\frac{1}{2}(a-b)_x \mathrm{d}z + \frac{1}{2}(a-b)_y \mathrm{d}t. \end{aligned} \tag{9}$$

Now the result follows from (9) and the well-known theorem [5,17] that an almost hyper-parahermitian structure is hyper-parahermitian if and only if the three Lee forms coincide. □

Remark 1. Observe that all Walker metrics defined by (8) are self-dual (see (14)) and moreover, the Ricci operator has a unique eigenvalue $\lambda = \frac{3}{2}\mathcal{B}(z,t)$, which is a double root of its minimal polynomial.

Formulas (9) also imply the following.

Theorem 3. *A proper almost hyper-parahermitian structure J_1, J_2, J_3 is hyper-parakählerian if and only if the functions a, b and c do not depend on x and y.*

It is well known [18] that, as in the definite case [5], any hyper-parakählerian structure is Ricci flat. Neutral Ricci flat non-flat metrics on complex tori and primary Kodaira surfaces have been constructed in [24]. These metrics are induced by proper hyper-parakählerian–Walker structures on $\mathbb{R}^4$. Further observe that proper hyper-parakähler structures correspond to Walker metrics admitting two orthogonal parallel null vector fields $\{\partial_x, \partial_y\}$.

4. Proper Hermitian–Walker structures

The existence of a metric of signature $(++--)$ with structure group $SO_0(2,2)$ is equivalent to the existence of a pair of commuting almost complex structures [20], and moreover, any such pseudo-Riemannian metric may be viewed as an indefinite almost Hermitian metric for a suitable almost complex structure. These almost complex structures are not uniquely determined. One such structure associated with a four-dimensional Walker metric in the form (4) was given in [21] and called the *proper* almost complex structure. Our purpose here is to investigate curvature properties of Walker metrics by considering the associated proper structure. It turns out that this structure exhibits a very rich behavior providing examples, as we mentioned above, of indefinite Ricci flat (non-flat) Kähler structures on tori and primary Kodaira surfaces [24] as well as flat non-Kähler almost Kähler structures [8]. This is in a sharp contrast to the Riemannian case and it is important to recognize that such an exceptional behavior comes from the fact that any proper almost Hermitian structure is isotropic Kähler but not necessarily Kähler.

The proper almost complex structure associated with the metric (4) coincides with the structure J_1 defined in Section 3 and is given by

$$\begin{aligned} J\partial_x &= \partial_y, \qquad J\partial_z = -c\partial_x + \frac{1}{2}(a-b)\partial_y + \partial_t, \\ J\partial_y &= -\partial_x, \qquad J\partial_t = \frac{1}{2}(a-b)\partial_x + c\partial_y - \partial_z. \end{aligned} \tag{10}$$

According to [21], the proper almost Hermitian structure (g, J) is:

- almost Kähler if and only if

$$a_x + b_x = 0, \qquad a_y + b_y = 0. \tag{11}$$

- Hermitian if and only if

$$a_x - b_x = 2c_y, \qquad a_y - b_y = -2c_x. \tag{12}$$

- Kähler if and only if

$$a_x = -b_x = c_y, \qquad a_y = -b_y = -c_x. \tag{13}$$

Self-dual Walker metrics have been previously investigated in [7,9] where it has been shown that a metric (4) is self-dual if and only if the functions a, b, c have the form

$$\begin{aligned} a(x,y,z,t) &= x^3\mathcal{A} + x^2\mathcal{B} + x^2y\mathcal{C} + xy\mathcal{D} + xP + yQ + \xi, \\ b(x,y,z,t) &= y^3\mathcal{C} + y^2\mathcal{E} + xy^2\mathcal{A} + xy\mathcal{F} + xS + yT + \eta, \\ c(x,y,z,t) &= \frac{1}{2}x^2\mathcal{F} + \frac{1}{2}y^2\mathcal{D} + x^2y\mathcal{A} + xy^2\mathcal{C} + \frac{1}{2}xy(\mathcal{B}+\mathcal{E}) + xU + yV + \gamma, \end{aligned} \tag{14}$$

where the capital, calligraphic and Greek letters stand for arbitrary smooth functions depending only on the coordinates (z, t).

Now it follows from (14) using (12) and (13) that

Theorem 4. *A proper Hermitian structure (g, J) on a Walker 4-manifold is self-dual if and only if*

$$\begin{aligned} a &= x^2\mathcal{B} + xy\mathcal{D} + xP + yQ + \xi, \\ b &= y^2\mathcal{B} - xy\mathcal{D} + xS + yT + \eta, \\ c &= \frac{1}{2}(y^2 - x^2)\mathcal{D} + xy\mathcal{B} - \frac{1}{2}x(Q-T) + \frac{1}{2}y(P-S) + \gamma, \end{aligned} \tag{15}$$

where all capital, calligraphic and Greek letters are arbitrary smooth functions depending only on the coordinates (z, t).

Remark 2. Note that the Ricci operator of any Walker metric (15) has complex eigenvalues $\lambda = \frac{3}{2}\mathcal{B} \pm \mathrm{i}\mathcal{D}$ of multiplicity two.

An indefinite Hermitian manifold (M, g, J) is called *locally conformally Kähler* if for any point $p \in M$ there exists an open neighborhood U and a function $f : U \to \mathbb{R}$ such that $(U, \mathrm{e}^{-f}g, J)$ is an indefinite Kähler manifold [10].

Theorem 5. *The structure (g, J) is locally conformally Kähler if and only if the functions a, b and c have the form*

$$\begin{aligned} a &= xP(z,t) + yQ(z,t) + \mathcal{A}(x,y,z,t) \\ b &= xP(z,t) + yQ(z,t) + \eta(z,t) - \mathcal{A}(x,y,z,t) \\ c &= \mathcal{B}(x,y,z,t), \end{aligned} \tag{16}$$

where P, Q and η are smooth functions depending only on the coordinates (z, t), the function $\mathcal{A} + \mathrm{i}\mathcal{B}$ is holomorphic with respect to $w = x + \mathrm{i}y$ and $P_t = Q_z$. Moreover, (g, J) is Kähler if and only if $P = Q = 0$.

Proof. Recall that a Hermitian structure (g, J) is locally conformally Kähler if and only if the Lee form ω defined by $\mathrm{d}\Omega = \omega \wedge \Omega$ is a closed 1-form [11]. Now, observe that the Lee form ω of the proper almost Hermitian structure (g, J) is given by

$$\omega = -\frac{1}{2}(a+b)_x\,\mathrm{d}z - \frac{1}{2}(a+b)_y\,\mathrm{d}t \tag{17}$$

(see the first equation in (9)). Therefore, $\mathrm{d}\omega = 0$ if and only if

$$(a+b)_{xx} = (a+b)_{xy} = (a+b)_{yy} = 0, \qquad (a+b)_{xt} = (a+b)_{yz}. \tag{18}$$

In particular, the function $a + b$ is linear with respect to x and y, i.e., it has the form

$$a + b = 2xP(z,t) + 2yQ(z,t) + \eta(z,t), \tag{19}$$

where P, Q and η are smooth functions. Moreover, since the almost complex structure J is integrable, we have by (12) and (19) that

$$(a - xP(z,t) - yQ(z,t))_x = c_y, \qquad (a - xP(z,t) - yQ(z,t))_y = -c_x,$$

which are the Cauchy–Riemann equations for the functions $\mathcal{A} = a - xP(z,t) - yQ(z,t)$ and $\mathcal{B} = c$ with respect to the variables x and y. Thus the function $\mathcal{A} + i\mathcal{B}$ is holomorphic with respect to $w = x + iy$ and a, b, c have the form (16). Also note that the equation $(a+b)_{xt} = (a+b)_{yz}$ implies $P_t = Q_z$. Finally, the Kähler condition follows from (13). □

Remark 3. The metrically equivalent vector field B of the Lee form ω is called the *Lee vector field* and JB is usually named as the *anti-Lee vector field*. A special class of locally conformally Kähler structures, the so-called *Vaisman manifolds*, corresponds to the case of parallel Lee form. Note that although proper locally conformally Kähler structures are not necessarily Vaisman, the distribution generated by B and JB is parallel since $B = -P(z,t)\partial_x - Q(z,t)\partial_y$. Further observe that many of the striking differences between the positive definite locally conformally Kähler structures and the indefinite counterpart lie on the fact that the distribution $\{B, JB\}$ may be degenerate, which indeed holds in the case under consideration (see [10]).

Theorem 6. *Let (g, J) be a proper Hermitian structure with nowhere vanishing Lee form ω. Then ω is parallel if and only if the functions a, b, c have the form*

$$\begin{aligned} a &= -2x\frac{PP_z + QQ_z}{P^2+Q^2} + 2y\frac{PQ_z - QP_z}{P^2+Q^2} + \xi \\ b &= 2x\frac{QP_t - PQ_t}{P^2+Q^2} - 2y\frac{PP_t + QQ_t}{P^2+Q^2} + \eta \\ c &= x\frac{Q(P_z - Q_t) - 2PQ_z}{P^2+Q^2} - y\frac{P(P_z - Q_t) + 2QQ_z}{P^2+Q^2} + \gamma, \end{aligned} \tag{20}$$

where P, Q, ξ, η, γ are smooth functions of (z,t) and

$$P_t - Q_z = 0, \qquad P_z + Q_t = -(P^2 + Q^2). \tag{21}$$

Proof. The Levi-Civita connection of a Walker metric (4) is determined by (see for example [9]):

$$\begin{aligned} \nabla_{\partial_x}\partial_z &= \frac{1}{2}a_x\partial_x + \frac{1}{2}c_x\partial_y, \qquad \nabla_{\partial_x}\partial_t = \frac{1}{2}c_x\partial_x + \frac{1}{2}b_x\partial_y, \\ \nabla_{\partial_y}\partial_z &= \frac{1}{2}a_y\partial_x + \frac{1}{2}c_y\partial_y, \qquad \nabla_{\partial_y}\partial_t = \frac{1}{2}c_y\partial_x + \frac{1}{2}b_y\partial_y, \\ \nabla_{\partial_z}\partial_z &= \frac{1}{2}(aa_x + ca_y + a_z)\partial_x + \frac{1}{2}(ca_x + ba_y - a_t + 2c_z)\partial_y - \frac{a_x}{2}\partial_z - \frac{a_y}{2}\partial_t, \\ \nabla_{\partial_z}\partial_t &= \frac{1}{2}(a_t + ac_x + cc_y)\partial_x + \frac{1}{2}(b_z + cc_x + bc_y)\partial_y - \frac{c_x}{2}\partial_z - \frac{c_y}{2}\partial_t, \\ \nabla_{\partial_t}\partial_t &= \frac{1}{2}(ab_x + cb_y - b_z + 2c_t)\partial_x + \frac{1}{2}(cb_x + bb_y + b_t)\partial_y - \frac{b_x}{2}\partial_z - \frac{b_y}{2}\partial_t. \end{aligned} \tag{22}$$

Straightforward computations making use of (17) and (22) show that the Lee form ω is parallel if and only if the structure (g, J) is locally conformally Kähler and the functions a, b, c satisfy the following equations

$$\begin{aligned} &a_x(a+b)_x + a_y(a+b)_y = -2(a+b)_{xz}, \qquad b_x(a+b)_x + b_y(a+b)_y = -2(a+b)_{yt}, \\ &c_x(a+b)_x + c_y(a+b)_y = -2(a+b)_{yz} = -2(a+b)_{xt}. \end{aligned}$$

In view of Theorem 5, the latter conditions are equivalent to the requirement that the functions $\mathcal{A}$ and $\mathcal{B}$ have the form

$$\mathcal{A} = \alpha x + \beta y + \xi, \qquad \mathcal{B} = -\beta x + \alpha y + \gamma,$$

where $\alpha, \beta, \xi, \gamma$ are functions of (z, t),

$$\alpha = -\frac{P(P_z - Q_t) + 2QQ_z}{P^2 + Q^2}, \qquad \beta = -\frac{Q(P_z - Q_t) - 2PQ_z}{P^2 + Q^2}$$

and P, Q satisfy (21). Now the result follows from (16). □

Example. A particular solution of (21) is given by

$$P = \frac{p}{pz + qt + r} \quad \text{and} \quad Q = \frac{q}{pz + qt + r},$$

where p, q, r are constants and $p^2 + q^2 \neq 0$. In this case

$$a = \frac{2px}{pz + qt + r} + \xi, \qquad b = \frac{2qy}{pz + qt + r} + \eta, \qquad c = \frac{qx + py}{pz + qt + r} + \gamma.$$

For the particular case of self-dual Walker metrics one has the following:

Theorem 7. *A proper Hermitian structure (g, J) on a Walker 4-manifold is locally conformally Kähler self-dual if and only if*

$$\begin{aligned} a &= xy\mathcal{D} + xP + yQ + \xi, \\ b &= -xy\mathcal{D} + xS + yT + \eta, \\ c &= \frac{1}{2}(y^2 - x^2)\mathcal{D} - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma, \end{aligned} \tag{23}$$

where all calligraphic and Greek letters are arbitrary smooth functions depending only on the coordinates (z, t) while capital letters depend only on the coordinates (z, t) and satisfy

$$(Q + T)_z = (P + S)_t. \tag{24}$$

Proof. Considering the characterization (18), the result is easily obtained imposing $(a + b)_{xx} = 0$ and $(a + b)_{xt} = (a + b)_{yz}$ on (15). □

Corollary 8. *A proper Hermitian structure (g, J) is Kähler self-dual if and only if*

$$\begin{aligned} a &= xy\mathcal{D} + xP + yQ + \xi, \\ b &= -xy\mathcal{D} - xP - yQ + \eta, \\ c &= \frac{1}{2}(y^2 - x^2)\mathcal{D} - xQ + yP + \gamma. \end{aligned} \tag{25}$$

5. ∗-Einstein-proper Hermitian structures

Associated to an almost Hermitian structure (g, J) we consider the ∗-Ricci tensor defined by $\rho^*(X, Y) = \operatorname{trace}\{U \rightsquigarrow -\frac{1}{2}JR(X, JY)U\}$ and the ∗-scalar curvature $\tau^* = \operatorname{trace}\rho^*$. Note that both ρ and ρ^* coincide in the Kähler setting but ρ^* is not symmetric in general. An n-dimensional almost Hermitian manifold (M, g, J) is called *weakly ∗-Einstein* if $\rho^* = \frac{\tau^*}{n}g$ and is said to be *∗-Einstein* if, in addition, τ^* is a constant.

Theorem 9. *The structure (g, J) is Hermitian and ∗-Einstein if and only if the functions a, b, c have one of the following three forms:*

$$\begin{aligned} a &= \kappa(x^2 - y^2) + xP + yQ + \xi, \\ b &= \kappa(y^2 - x^2) - xP - yQ - \xi + \frac{1}{\kappa}(P_z - Q_t), \\ c &= 2\kappa xy - xQ + yP + \gamma, \end{aligned} \tag{26}$$

or

$$\begin{aligned} a &= \kappa x^2 + xP + yQ + \xi, \\ b &= \kappa y^2 + xS + yT - \xi - \frac{1}{4\kappa}\{4(3S_z - P_z + 3Q_t - T_t) - (P+S)^2 - (Q+T)^2\}, \\ c &= \kappa xy - \frac{1}{2}x(Q-T) + \frac{1}{2}y(P-S) + \gamma, \end{aligned} \tag{27}$$

or

$$\begin{aligned} a &= xP + yQ + \xi, \\ b &= xS + yT + \eta, \\ c &= -\frac{1}{2}x(Q-T) + \frac{1}{2}y(P-S) + \gamma, \end{aligned} \tag{28}$$

where in the last case

$$4(3S_z - P_z + 3Q_t - T_t) = (P+S)^2 + (Q+T)^2. \tag{29}$$

Here κ is a non-zero constant and capital and Greek letters are smooth functions depending only on (z,t). In the first case $\tau^* = 8\kappa$, in the second one $\tau^* = 2\kappa$ and in the third case $\tau^* = 0$.

Proof. The $*$-Einstein equation ($\rho_0^* = \rho^* - \frac{\tau^*}{4}g = 0$) for a proper almost Hermitian structure (g, J) defined by (4) and (10) can be written as a system of PDEs as follows (we refer to [9,21] for the curvature formulas to determine the $*$-Ricci tensor):

$$\begin{aligned} (\rho_0^*)_{xz} &= -(\rho_0^*)_{yt} = -(\rho_0^*)_{zx} = (\rho_0^*)_{ty} = \frac{1}{4}(a_{yy} - b_{xx}) = 0, \\ (\rho_0^*)_{xt} &= -(\rho_0^*)_{zy} = -\frac{1}{2}(a_{xy} - c_{xx}) = 0, \\ (\rho_0^*)_{yz} &= -(\rho_0^*)_{tx} = -\frac{1}{2}(b_{xy} - c_{yy}) = 0, \\ (\rho_0^*)_{zz} &= \frac{1}{4}\Big\{a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2 + 2c(a_{xy} - c_{xx}) \\ &\quad + b a_{yy} - 2a_{yt} + a b_{xx} - 2b_{xz} - (a+b)c_{xy} + 2c_{xt} + 2c_{yz}\Big\} = 0, \\ (\rho_0^*)_{zt} &= -\frac{1}{4}\{(a-b)(a_{xy} - c_{xx}) + c(a_{yy} - b_{xx})\} = 0, \\ (\rho_0^*)_{tz} &= \frac{1}{4}\{(a-b)(b_{xy} - c_{yy}) + c(a_{yy} - b_{xx})\} = 0, \\ (\rho_0^*)_{tt} &= \frac{1}{4}\Big\{a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2 + 2c(b_{xy} - c_{yy}) \\ &\quad + b a_{yy} - 2a_{yt} + a b_{xx} - 2b_{xz} - (a+b)c_{xy} + 2c_{xt} + 2c_{yz}\Big\} = 0. \end{aligned} \tag{30}$$

Note that the $*$-scalar curvature is given by

$$\tau^* = -a_{yy} - b_{xx} + 2\,c_{xy}. \tag{31}$$

If follows from (12) and (30) that (g, J) is Hermitian and weakly $*$-Einstein if and only if

$$a_x - b_x = 2c_y, \qquad a_y - b_y = -2c_x, \qquad a_{xy} = c_{xx}, \qquad b_{xy} = c_{yy}, \qquad a_{yy} = b_{xx}, \tag{32}$$

and

$$\begin{aligned} &a_x b_x + a_y b_y - a_y c_x + b_y c_x + a_x c_y - b_x c_y - c_x^2 - c_y^2 \\ &\quad + b a_{yy} - 2a_{yt} + a b_{xx} - 2b_{xz} - (a+b)c_{xy} + 2c_{xt} + 2c_{yz} = 0. \end{aligned} \tag{33}$$

As an immediate consequence from (32) we get $a_{xy} = b_{xy} = c_{xx} = c_{yy} = 0$ and therefore

$$\begin{aligned} a &= A(x, z, t) + \bar{A}(y, z, t), \\ b &= B(y, z, t) + \bar{B}(x, z, t), \\ c &= xyC(z, t) + xU(z, t) + yV(z, t) + \gamma(z, t). \end{aligned} \tag{34}$$

Now, $a_{yy} = b_{xx}$ in (32) means $\bar{A}_{yy}(y, z, t) = \bar{B}_{xx}(x, z, t)$. Hence

$$\bar{A}(y, z, t) = \frac{1}{2}y^2 D(z, t) + yQ(z, t) + \bar{\xi}(z, t), \qquad \bar{B}(x, z, t) = \frac{1}{2}x^2 D(z, t) + xS(z, t) + \bar{\eta}(z, t), \tag{35}$$

and $a_x - b_x = 2c_y$, $a_y - b_y = -2c_x$ in (32) lead to

$$\begin{aligned} A(x, z, t) &= \frac{1}{2}x^2(2C(z, t) + D(z, t)) + x(S(z, t) + 2V(z, t)) + \tilde{\xi}(z, t), \\ B(y, z, t) &= \frac{1}{2}y^2(2C(z, t) + D(z, t)) + y(Q(z, t) + 2U(z, t)) + \tilde{\eta}(z, t). \end{aligned} \tag{36}$$

Collecting together (34)–(36) and setting $2L = 2C + D$, $M = \frac{1}{2}D$, $P = S + 2V$, $T = Q + 2U$, $\xi = \bar{\xi} + \tilde{\xi}$, $\eta = \bar{\eta} + \tilde{\eta}$ we obtain

$$\begin{aligned} a &= x^2 L + y^2 M + xP + yQ + \xi, \\ b &= x^2 M + y^2 L + xS + yT + \eta, \\ c &= xy(L - M) - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma. \end{aligned} \tag{37}$$

Now, using (31), the (constant) $*$-scalar curvature is given by $\tau^* = 2(L - 3M)$, from where

$$M = \frac{1}{6}(2L - \tau^*).$$

Plugging a, b, c in (33) and differentiating twice by x we have

$$(\tau^*)^2 - 10\tau^* L(z, t) + 16L(z, t)^2 = 0.$$

Thus, $L(z, t) = \kappa$ must be a constant, and

$$\begin{aligned} a &= x^2\kappa + \frac{1}{6}y^2(2\kappa - \tau^*) + xP + yQ + \xi, \\ b &= \frac{1}{6}x^2(2\kappa - \tau^*) + y^2\kappa + xS + yT + \eta, \\ c &= \frac{1}{6}xy(4\kappa + \tau^*) - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma, \end{aligned} \tag{38}$$

where $\kappa = \frac{\tau^*}{8} \neq 0$, $\kappa = \frac{\tau^*}{2} \neq 0$ or $\kappa = \tau^* = 0$. The proof finishes by analyzing each of these cases separately.

Starting with the case $\kappa = \frac{\tau^*}{8} \neq 0$ and differentiating (33) by x and by y we have

$$\tau^*(P + S) = 0, \qquad \tau^*(Q + T) = 0,$$

which implies $S = -P$ and $T = -Q$. Thus, (33) reduces to $\tau^*(\xi + \eta) - 8(P_z - Q_t) = 0$, from where

$$\eta = \frac{1}{\kappa}(P_z - Q_t) - \xi$$

and (26) is obtained from (38). Now, if $\kappa = \frac{\tau^*}{2}$, (33) reduces to

$$2\tau^*(\xi + \eta) + 4(3S_z - P_z + 3Q_t - T_t) = (Q + T)^2 + (P + S)^2. \tag{39}$$

Hence, for $\tau^* \neq 0$, this last equation determines η and (27) is obtained from (38). Otherwise, for $\tau^* = 0$, (38) reduces to (28) and (39) to (29), finishing the proof. □

Remark 4. Metrics (28) can be viewed as *Riemann extensions*, i.e. they are locally isometric to the cotangent bundle $(T^*\Sigma, g_D + \pi^*\phi)$ of a torsion-free affine surface (Σ, D) equipped with the metric $g_D(X^C, Y^C) = -\iota(D_X Y + D_Y X)$, where X^C and Y^C are complete lifts to the cotangent bundle, $\iota Z^k \partial_k = x_{k'} Z^k$ for any vector field $Z = Z^k \partial_k$, $\pi^*\phi$ is the pull-back on $T^*\Sigma$ of a symmetric (0, 2)-tensor field on Σ, and $(x_k, x_{k'})$ are natural coordinates on $T^*\Sigma$ induced by coordinates (x_k) on Σ. (See [6,28] and the references therein for more information on the geometry of Riemann extensions.)

Theorem 10. *The structure (g, J) is strictly locally conformally Kähler and $*$-Einstein if and only if the functions a, b and c have the form*

$$\begin{aligned}
a &= xP(z,t) + yQ(z,t) + \xi(z,t),\\
b &= xS(z,t) + yT(z,t) + \eta(z,t),\\
c &= -\frac{1}{2}x(Q(z,t) - T(z,t)) + \frac{1}{2}y(P(z,t) - S(z,t)) + \gamma(z,t),
\end{aligned} \tag{40}$$

where at least one of $P + S$ and $Q + T$ does not vanish everywhere and

$$4(3S_z - P_z + 3Q_t - T_t) = (P + S)^2 + (Q + T)^2, \qquad (Q + T)_z = (P + S)_t. \tag{41}$$

Proof. We analyze the three families of metrics obtained in Theorem 9 using the characterization (18). First note that metrics (26) are Kähler. For metrics (27), $(a + b)_{xx} = 2\kappa \neq 0$, and therefore the locally conformally Kähler condition does not hold. Finally, for metrics (28), one easily checks that the locally conformally Kähler condition for this family of Hermitian–$*$-Einstein–Walker metrics is equivalent to $(Q + T)_z = (P + S)_t$. Moreover, these metrics are Kähler if and only if $P + S = Q + T = 0$ (see (13)), from where the result follows. □

Remark 5. A proper almost Hermitian structure (g, J) is Hermitian, self-dual and $*$-Einstein if and only if the functions a, b and c are given by (27) or (28) in Theorem 9.

Remark 6. If the function c depends only on (z, t), then the structure (g, J) is Hermitian and $*$-Einstein if and only if the functions a and b have the form

$$\begin{aligned}
a &= xP(z,t) + yQ(z,t) + \xi(z,t),\\
b &= xP(z,t) + yQ(z,t) + \eta(z,t),
\end{aligned} \tag{42}$$

where

$$2(P_z + Q_t) = P^2 + Q^2. \tag{43}$$

A particular solution of (43) is given by

$$P = -\frac{2}{z + f(t)}, \qquad Q = -\frac{2}{t + g(z)},$$

where $f(t)$ and $g(z)$ are smooth functions. According to Remark 6 the structure (g, J) is Hermitian and $*$-Einstein, but it is neither Einstein nor locally conformally Kähler in general.

6. Einstein-proper Hermitian structures

The Einstein equation ($\rho_0 = \rho - \frac{\tau}{4}\, g = 0$) for a Walker metric (4) is a system of PDEs as follows (cf. [21]):

$$
\begin{aligned}
&(\rho_0)_{xz} = -(\rho_0)_{yt} = (\rho_0)_{zx} = -(\rho_0)_{ty} = \frac{1}{4}\,(a_{xx} - b_{yy}) = 0,\\
&(\rho_0)_{xt} = (\rho_0)_{tx} = \frac{1}{2}\,(b_{xy} + c_{xx}) = 0,\\
&(\rho_0)_{yz} = (\rho_0)_{zy} = \frac{1}{2}\,(a_{xy} + c_{yy}) = 0,\\
&(\rho_0)_{zt} \;=\; (\rho_0)_{tz} = \frac{1}{4}\,\{-2\,a_y\,b_x + 2\,c_x\,c_y - c\,a_{xx} + 2\,a_{xt} - c\,b_{yy}\\
&\qquad\qquad + 2\,b_{yz} + 2\,a\,c_{xx} + 2\,c\,c_{xy} - 2\,c_{xz} + 2\,b\,c_{yy} - 2\,c_{yt}\} = 0,\\
&(\rho_0)_{zz} \;=\; \frac{1}{4}\,\Big\{2\,a_x\,c_y + 2\,a_y\,b_y - 2\,a_y\,c_x - 2\,c_y^2 + a\,a_{xx} + 4\,c\,a_{xy}\\
&\qquad\qquad + 2\,b\,a_{yy} - 4\,a_{yt} - a\,b_{yy} - 2\,a\,c_{xy} + 4\,c_{yz}\Big\} = 0,\\
&(\rho_0)_{tt} \;=\; \frac{1}{4}\,\Big\{2\,a_x\,b_x - 2\,b_x\,c_y + 2\,b_y\,c_x - 2\,c_x^2 - b\,a_{xx} + 2\,a\,b_{xx}\\
&\qquad\qquad + 4\,c\,b_{xy} - 4\,b_{xz} + b\,b_{yy} - 2\,b\,c_{xy} + 4\,c_{xt}\Big\} = 0.
\end{aligned}
\tag{44}
$$

Note that the scalar curvature is given by

$$\tau = a_{xx} + b_{yy} + 2\,c_{xy}. \tag{45}$$

Theorem 11. *The structure (g, J) is Hermitian–Einstein if and only if the functions a, b, c have one of the following three forms:*

$$
\begin{aligned}
&a = \kappa(x^2 - y^2) + xP + yQ + \xi,\\
&b = \kappa(y^2 - x^2) - xP - yQ - \xi + \frac{1}{\kappa}(P_z - Q_t),\\
&c = 2\kappa xy - xQ + yP + \gamma,
\end{aligned}
\tag{46}
$$

or

$$
\begin{aligned}
&a = \kappa x^2 + xP + yQ + \frac{1}{4\kappa}\{4(P - S)_z - 8Q_t + 2Q(Q + T) + (P^2 - S^2)\},\\
&b = \kappa y^2 + xS + yT + \frac{1}{4\kappa}\{-4(Q - T)_t - 8S_z + 2S(P + S) - (Q^2 - T^2)\},\\
&c \;=\; \kappa xy - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S)\\
&\qquad + \frac{1}{4\kappa}\{2(P + S)_t + 2(Q + T)_z + T(P - S) - Q(P + 3S)\},
\end{aligned}
\tag{47}
$$

or

$$
\begin{aligned}
&a = xP + yQ + \xi,\\
&b = xS + yT + \eta,\\
&c = -\frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma,
\end{aligned}
\tag{48}
$$

where in the last case

$$\begin{aligned} &8Q_t - 4(P-S)_z = 2Q(Q+T) + (P^2 - S^2),\\ &8S_z + 4(Q-T)_t = 2S(P+S) - (Q^2 - T^2),\\ &2(P+S)_t + 2(Q+T)_z = Q(P+3S) - T(P-S). \end{aligned} \tag{49}$$

Here κ is a non-zero constant and capital and Greek letters are smooth functions depending only on (z,t). In the first case $\tau = 8\kappa$, in the second one $\tau = 6\kappa$ and in the third case $\tau = 0$.

Remark 7. Metrics (48) and (49) have received attention in the literature of Osserman manifolds. Since any four-dimensional Riemann extension is self-dual [6], metrics (48) and (49) are Osserman with nilpotent Jacobi operators and moreover they are realized as Riemann extensions of torsion-free connections with skew-symmetric Ricci tensor [9,12].

Proof. If follows from (12) and the first three equations in (44) that (g,J)-Hermitian–Einstein implies

$$a_x - b_x = 2c_y, \qquad a_y - b_y = -2c_x, \qquad a_{xy} = -c_{yy}, \qquad b_{xy} = -c_{xx}, \qquad a_{xx} = b_{yy}. \tag{50}$$

Hence, we get $a_{xy} = b_{xy} = c_{xx} = c_{yy} = 0$ and therefore

$$\begin{aligned} &a = A(y,z,t) + \bar{A}(x,z,t),\\ &b = B(x,z,t) + \bar{B}(y,z,t),\\ &c = xyC(z,t) + xU(z,t) + yV(z,t) + \gamma(z,t). \end{aligned} \tag{51}$$

Now, $a_{xx} = b_{yy}$ in (50) means $\bar{A}_{xx}(x,z,t) = \bar{B}_{yy}(y,z,t)$. Hence

$$\bar{A}(x,z,t) = \frac{1}{2}x^2 D(z,t) + xP(z,t) + \bar{\xi}(z,t), \qquad \bar{B}(y,z,t) = \frac{1}{2}y^2 D(z,t) + yT(z,t) + \bar{\eta}(z,t). \tag{52}$$

Moreover, $a_x - b_x = 2c_y$, $a_y - b_y = -2c_x$ in (50) lead to

$$\begin{aligned} &A(y,z,t) = \frac{1}{2}y^2(D(z,t) - 2C(z,t)) + y(T(z,t) - 2U(z,t)) + \tilde{\xi}(z,t),\\ &B(x,z,t) = \frac{1}{2}x^2(D(z,t) - 2C(z,t)) + x(P(z,t) - 2V(z,t)) + \tilde{\eta}(z,t). \end{aligned} \tag{53}$$

Collecting together (51)–(53) and setting $2M = D - 2C$, $L = \frac{1}{2}D$, $Q = T - 2U$, $S = P - 2V$, $\xi = \bar{\xi} + \tilde{\xi}$, $\eta = \bar{\eta} + \tilde{\eta}$ we obtain

$$\begin{aligned} &a = x^2 L + y^2 M + xP + yQ + \xi,\\ &b = x^2 M + y^2 L + xS + yT + \eta,\\ &c = xy(L-M) - \frac{1}{2}x(Q-T) + \frac{1}{2}y(P-S) + \gamma. \end{aligned} \tag{54}$$

Now, the (constant) scalar curvature is given by $\tau = 2(3L - M)$, from where

$$M = \frac{1}{2}(6L - \tau).$$

Then, plugging a, b, c in the fourth equation of (44) and differentiating twice by x, y we get

$$\tau^2 - 14\tau L(z,t) + 48L(z,t)^2 = 0,$$

which implies that $L(z,t) = \kappa$ must be a constant and

$$\begin{aligned} &a = x^2\kappa + \frac{1}{2}y^2(6\kappa - \tau) + xP + yQ + \xi,\\ &b = \frac{1}{2}x^2(6\kappa - \tau) + y^2\kappa + xS + yT + \eta,\\ &c = \frac{1}{2}xy(\tau - 4\kappa) - \frac{1}{2}x(Q-T) + \frac{1}{2}y(P-S) + \gamma, \end{aligned} \tag{55}$$

where $\kappa = \frac{\tau}{8} \neq 0$, $\kappa = \frac{\tau}{6} \neq 0$ or $\kappa = \tau = 0$. Next we analyze each of these cases separately.

For $\kappa = \frac{\tau}{8} \neq 0$, we differentiate the last equation in (44) by x and by y, and get

$$\tau(P+S) = 0, \qquad \tau(Q+T) = 0,$$

and therefore $S = -P$ and $T = -Q$. With these conditions, a further analysis shows that (44) holds if and only if

$$\eta = \frac{1}{\kappa}(P_z - Q_t) - \xi,$$

from where (55) reduces to (46).

Now, if $\kappa = \frac{\tau}{6}$, the last three equations in (44) transform into

$$\begin{aligned} \frac{2\tau}{3}\xi &= 4(P-S)_z - 8Q_t + 2Q(Q+T) + (P^2 - S^2), \\ \frac{2\tau}{3}\eta &= -4(Q-T)_t - 8S_z + 2S(P+S) - (Q^2 - T^2), \\ \frac{2\tau}{3}\gamma &= 2(P+S)_t + 2(Q+T)_z + T(P-S) - Q(P+3S). \end{aligned} \tag{56}$$

Hence, for $\tau \neq 0$, these three equations determine ξ, η and γ and (47) is obtained from (55). Otherwise, for $\tau = 0$, (55) reduces to (48) and (56) to (49), showing the result. □

Corollary 12. *Any proper Hermitian–Einstein structure is ∗-Einstein.*

Remark 8. A proper almost Hermitian structure (g, J) is Hermitian, self-dual and Einstein if and only if the functions a, b and c are given by (47) or (48) in Theorem 11.

Remark 9. In the case when the function c depends only on (z, t), the structure (g, J) is Hermitian and Einstein if and only if the functions a and b have the form

$$\begin{aligned} a &= xP(z,t) + yQ(z,t) + \xi(z,t), \\ b &= xP(z,t) + yQ(z,t) + \eta(z,t), \end{aligned} \tag{57}$$

where

$$2P_z = P^2, \qquad 2Q_t = Q^2, \qquad P_t + Q_z = PQ. \tag{58}$$

Moreover, in a neighborhood of a point where $P^2 + Q^2 \neq 0$ the solution of the system (58) is given by

$$P = -\frac{2p}{pz + qt + r}, \qquad Q = -\frac{2q}{pz + qt + r}, \tag{59}$$

where p, q, r are constants and $p^2 + q^2 \neq 0$. Indeed, suppose that $P^2 + Q^2 \neq 0$ in a neighborhood of a point (z_0, t_0). Then the first two equations in (58) imply

$$P(z,t) = -\frac{2}{z + f(t)}, \qquad Q(z,t) = -\frac{2}{t + g(z)}, \tag{60}$$

where $f(t)$ and $g(z)$ are smooth functions, and the third equation in (58) takes the form

$$\frac{f_t}{(z + f(t))^2} + \frac{g_z}{(t + g(z))^2} = \frac{2}{(z + f(t)(t + g(z)))}. \tag{61}$$

Setting $F(t) = \frac{1}{z_0 + f(t)}$ and $\alpha = g(z_0)$, $\beta = g_z(z_0)$, the latter equation implies that

$$F_t + \frac{2F}{t+\alpha} = \frac{\beta}{(t+\alpha)^2}.$$

The solution of this linear ODE is given by

$$F(t) = \frac{\beta}{t+\alpha} + \frac{\gamma}{(t+\alpha)^2}, \quad \gamma = const.$$

Thus

$$f(t)=\frac{(t+\alpha)^2}{\gamma+\beta(t+\alpha)}.$$

In the same way we see that the function $g(z)$ has the form

$$g(z)=\frac{(z+\lambda)^2}{\mu+\nu(z+\lambda)}.$$

Substituting these expressions of f and g into (61), clearing the denominators and comparing the coefficients of the obtained polynomials of z and t, we see from (60) that P and Q have the form (59).

Note that if $p\cdot q\neq 0$, then the structure (g,J) is strictly locally conformally Kähler, self-dual, Ricci flat and $*$-Ricci flat, but the metric g is not flat.

Theorem 13. *The structure (g,J) is strictly locally conformally Kähler–Einstein if and only if the functions a, b and c have the form*

$$\begin{aligned}
a&=xP(z,t)+yQ(z,t)+\xi(z,t),\\
b&=xS(z,t)+yT(z,t)+\eta(z,t),\\
c&=-\frac{1}{2}x(Q(z,t)-T(z,t))+\frac{1}{2}y(P(z,t)-S(z,t))+\gamma(z,t),
\end{aligned}\tag{62}$$

where at least one of the functions $P+S$ and $Q+T$ does not vanish everywhere and

$$\begin{aligned}
&8Q_t-4(P-S)_z=2Q(Q+T)+(P^2-S^2),\\
&8S_z+4(Q-T)_t=2S(P+S)-(Q^2-T^2),\\
&4(P+S)_t=4(Q+T)_z=Q(P+3S)-T(P-S).
\end{aligned}\tag{63}$$

Proof. We analyze the three families of metrics obtained in Theorem 11 using the characterization (18). First note that metrics of the type (46) are Kähler (see (13)). For metrics (47), $(a+b)_{xx}=\frac{\tau}{3}\neq 0$, and therefore the locally conformally Kähler condition fails. Finally, for metrics (48), the locally conformally Kähler condition is equivalent to $(Q+T)_z=(P+S)_t$. Moreover, these metrics are Kähler if and only if $P+S=Q+T=0$, from where the result follows. □

Using (13), we easily obtain from Theorem 11 the following.

Corollary 14. *The structure (g,J) is Kähler–Einstein if and only if the functions a, b, c have one of the following forms*

$$\begin{aligned}
a&=\kappa(x^2-y^2)+xP+yQ+\xi,\\
b&=\kappa(y^2-x^2)-xP-yQ-\xi+\frac{1}{\kappa}(P_z-Q_t),\\
c&=2\kappa xy-xQ+yP+\gamma,
\end{aligned}\tag{64}$$

or

$$\begin{aligned}
a&=xP(z,t)+yQ(z,t)+\xi(z,t),\\
b&=-xP(z,t)-yQ(z,t)+\eta(z,t),\\
c&=-xQ(z,t)+yP(z,t)+\gamma(z,t),
\end{aligned}\tag{65}$$

where in the last case

$$P_z=Q_t.\tag{66}$$

Here κ is a non-zero constant and capital and Greek letters are smooth functions depending only on (z,t). In the first case $\tau=8\kappa$, in the second one $\tau=0$.

7. An example

It is proved in [19, Theorem 3.2] that an almost Hermitian 4-manifold (M, g, J) with positive definite metric is Hermitian if the metric g is Einstein and its positive Weyl curvature tensor satisfies the condition $\|W^+\|^2 = \frac{1}{24}(\tau - 3\tau^*)^2 \neq 0$ at every point of M. We shall give an example showing that the analog of this result is not true in the setting of signature $(2, 2)$. For metrics of this signature the role of W^+ is played by W^- since the almost complex structures compatible with the metric and the orientation are parametrized by sections of Λ_-. Also note that the metric on Λ^2 in [19] is one-half of the metric used here. Thus, in our situation, the analog of the above condition for W^+ is $\|W^-\|^2 = \frac{1}{96}(\tau - 3\tau^*)^2 \neq 0$.

Let g be the Walker metric on $\mathbb{R}^4$ for which

$$a = x^2 + \frac{2}{3}y^2 + \frac{2}{\sqrt{3}}xy, \quad b = y^2, \quad c = -\frac{1}{\sqrt{3}}y^2.$$

Then it follows from (44), (45) and (31) that the metric g is Einstein with $\tau = 4$ and $\tau^* = -\frac{4}{3}$. Moreover, using the formula for the components of W^- with respect to the basis (6) given in [9] we get

$$W^- = \begin{bmatrix} -\frac{2}{3} & \frac{1}{\sqrt{3}} & \frac{1}{3} \\ -\frac{1}{\sqrt{3}} & \frac{2}{3} & \frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

Therefore $\|W^-\|^2 = \frac{2}{3} = \frac{1}{96}(\tau - 3\tau^*)^2$. On the other hand, it follows from (12) that the proper almost complex structure J is not integrable. Note that W^- has degenerate spectrum $\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\}$. Moreover, the eigenspace corresponding to the simple eigenvalue is time-like (and hence it determines an integrable almost product structure [16]), thus showing that there is no integrable almost Hermitian structure in this example (see the work at [1,3]).

Acknowledgment

The second, third and the sixth named authors are supported by projects MTM2006-01432 and PGIDIT06PXIB207054PR.

References

[1] V. Apostolov, Generalized Goldberg–Sacks theorems for pseudo-Riemannian four-manifolds, J. Geom. Phys. 27 (1998) 185–198.
[2] V. Apostolov, T. Draghici, The curvature and the integrability of almost Kähler manifold: A survey, in: Symplectic and Contact Topology: Interactions and Perspectives, Toronto, ON/Montreal, QC, 2001, in: Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, pp. 25–53.
[3] V. Apostolov, P. Gauduchon, The Riemann Goldberg–Sacks theorems, Internat. J. Math. 8 (1997) 421–439.
[4] D. Blair, J. Davidov, O. Muškarov, Isotropic Kähler hyperbolic twistor spaces, J. Geom. Phys. 52 (2004) 74–88.
[5] C.P. Boyer, A note on hyperhermitian four-manifolds, Proc. Amer. Math. Soc. 102 (1988) 157–164.
[6] E. Calviño-Louzao, E. García-Río, R. Vázquez-Lorenzo, Riemann extensions of torsion-free connections with degenerate Ricci tensor (in press).
[7] J. Davidov, O. Muškarov, Self-dual Walker metrics with a two-step nilpotent Ricci operator, J. Geom. Phys. 57 (2006) 157–165.
[8] J. Davidov, J.C. Díaz-Ramos, E. García-Río, Y. Matsushita, O. Muškarov, R. Vázquez-Lorenzo, Almost Kähler Walker 4-manifolds, J. Geom. Phys. 57 (2007) 1075–1088.
[9] J.C. Díaz-Ramos, E. García-Río, R. Vázquez-Lorenzo, Four-dimensional Osserman metrics with nondiagonalizable Jacobi operators, J. Geom. Anal. 16 (2006) 39–52.
[10] S. Dragomir, K.L. Duggal, Indefinite locally conformal Kähler manifolds, Differential Geom. Appl. 25 (2007) 8–22.
[11] S. Dragomir, L. Ornea, Locally Conformally Kähler Geometry, in: Progress in Math., vol. 155, Birkhäuser, Boston, 1998.
[12] E. García-Río, D.N. Kupeli, M.E. Vázquez-Abal, R. Vázquez-Lorenzo, Affine Osserman connections and their Riemann extensions, Differential Geom. Appl. 11 (1999) 145–153.
[13] E. García-Río, Y. Matsushita, Isotropic Kähler structures on Engel 4-manifolds, J. Geom. Phys. 33 (2000) 288–294.
[14] S. Goldberg, Integrability of almost Kähler manifolds, Proc. Amer. Math. Soc. 21 (1969) 96–100.
[15] N. Hitchin, Hypersymplectic quotients, Acta Acad. Sci. Tauriensis 124 (1990) 169–180.

[16] S. Ivanov, S. Zamkovoy, Parahermitian and paraquaternionic manifolds, Differential Geom. Appl. 23 (2005) 205–234.
[17] H. Kamada, Self-duality of neutral metrics on four-manifolds, in: The Third Pacific Rim Geometry Conference, Seul, 1996, in: Monogr. Geom. Topology, vol. 25, Internat. Press, Cambridge, MA, 1998, pp. 79–98.
[18] H. Kamada, Neutral hyperkähler structures on primary Kodaira surfaces, Tsukuba J. Math. 23 (1999) 321–332.
[19] K.-D. Kirchberg, Integrability conditions for almost Hermitian and almost Kähler 4-manifolds. arXiv:math.DG/0605611.
[20] Y. Matsushita, Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds, Math. Z. 207 (1991) 281–291.
[21] Y. Matsushita, Walker 4-manifolds with proper almost complex structures, J. Geom. Phys. 55 (2005) 385–398.
[22] Y. Matsushita, S. Haze, P.R. Law, Almost Kähler–Einstein structures on 8-dimensional Walker manifolds, Monatsh. Math. 150 (2007) 41–48.
[23] P. Nurowski, Einstein equations and Cauchy–Riemann geometry, Doctor Philosophiae Thesis, SISSA, Trieste, 1993.
[24] J. Petean, Indefinite Kähler–Einstein Metrics on compact complex surfaces, Commun. Math. Phys. 189 (1997) 227–235.
[25] M. Przanowski, B. Broda, Locally Kähler gravitational instantons, Acta Phys. Polon. B 14 (1983) 637–661.
[26] K. Sekigawa, On some compact Einstein almost Kähler manifolds, J. Math. Soc. Japan 39 (1987) 677–684.
[27] A.G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, Quart. J. Math. Oxford (2) 1 (1950) 69–79.
[28] K. Yano, S. Ishihara, Tangent and Cotangent Bundles, Marcel Dekker, New York, 1973.